Average Case Complexity

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NP Problems

NP problem is
a (Ptime) set (of instances)
with a size function $|x|$ and
with a set of positive instances

Example:
HAMILTONIAN CIRCUIT.
Instances are graphs,
size = number of vertices,
a positive instance has
    a Hamiltonian circuit

DNP Problems

Distributional NP Problem
is an NP problem
with a probability distribution

Example:
HC with Uniform Distribution
so that
same size graphs are equally probable,
$\mathbf{P}$(exactly n vertices) $= \frac{1}{n(n+1)}$

Average Case may be Trivial even if Worst Case is Hard

Furthermore, a standard algorithm may do well on average.

Recall that
NP Complete (NPC) means
worst-case intractable

Example [Wilf].
Backtracking solves
3-Coloring with uniform distribution
in expected ?? time [Wilf].
Why?

Average Case May Have a Fast Algorithm that is Hard to Find

Example.

Hamiltonian Circuit (G-Shelah).

First algorithm:

quick and dirty,

looks only for a circuit.

Second algorithm:

sophisticated,

seeks a circuit or evidence

that there isn't any.

Third algorihm:

an unusual exhaustive search.

Morals

1. Think probabilities.
Fit your algorithm
not only to the NP problem
but also to the prob. distr.

Example. Improve Wilf's result.

2. Standard algorithms
may not work and thus
new algorithms may be needed.

3. A cascade of algorithms
may be needed.

4. Internal randomization
may help.

Toward Negative Results

The importance of
negative results in general and
NP completeness in particular

What's "polynomial on average"?
Let $\mathbf{P}_n(x) = \mathbf{P}(x \mid |x| = n)$.
The obvious definition
(1) $(\exists p)(\forall n)\Big[\sum T(x)\mathbf{P}_n(x) \leq p(n)\Big]$
or
(2) $(\exists k, c)(\forall n)\Big[\sum \frac{T(x)}{n^k}\mathbf{P}_n(x) \leq c\Big]$.
does not work.
Leonid Levin's Solution:
(3) $\mathbf{E}(T(x)^\epsilon/|x|) < \infty$
for some $\epsilon > 0$.

A decision problem is APtime
if it admits a decision algorithm
that runs in AP time.

Average-Case Complete Problems

Why standard reductions do not work?

See complete problems and much more
at Average-Case Complexity Forum,
http://www.uncg.edu/mat/avg.html

The utility of artificial
complete problems

The importance of internal randomization

Challenger and Solver